Announcements

1) Added corollary to previous day

2) Candidate talk. Doug LaFountain

3) If $\lim_{n\to\infty} a_n = L$ and

$\lim_{n\to\infty} b_n = M$, then

$$\lim_{n\to\infty} a_n b_n = LM.$$

proof: Let $\varepsilon > 0$ we want

$$|a_n b_n - LM| < \varepsilon$$

Use triangle inequality

$$|a_n b_n - L b_n + L b_n - LM|$$

$$= |a_n b_n - LM|$$

$$|a_n b_n - L b_n + L b_n - LM|$$

$$\leq |a_n b_n - L b_n| + |L b_n - LM|$$

$$= |b_n||a_n - L| + |L|\,|\underbrace{b_n - M}|$$

what about this term?

can make this $< \dfrac{\varepsilon}{|L|}$

Choose $N_1 \in \mathbb{N}$ so that

$$|b_n - M| < \frac{\varepsilon}{2|L|} \quad \forall\, n \geq N_1.$$

We then have for all such n,

$$|b_n||a_n - L| + |L||b_n - M|$$

$$< |b_n||a_n - L| + \cancel{|L|} \cdot \frac{\varepsilon}{2\cancel{|L|}}$$

(assuming $L \neq 0$)

$$= |b_n||a_n - L| + \frac{\varepsilon}{2}$$

But since $|b_n - M| < \frac{\varepsilon}{2|L|}$,

and $|b_n - M| \geq ||b_n| - |M||$,

we have

$$|b_n| - |M| < \frac{\varepsilon}{2|L|}, \text{ so}$$

$$|b_n| < |M| + \frac{\varepsilon}{2|L|}$$

Hence for all $n \geq N_1$,

$$|b_n|\,|a_n - L| + \frac{\varepsilon}{2}$$

$$< \left(|M| + \frac{\varepsilon}{2|L|}\right)|a_n - L| + \frac{\varepsilon}{2}$$

Choose $N_2 \in \mathbb{N}$ so that

$$|a_n - L| < \frac{\varepsilon}{2\left(|M| + \frac{\varepsilon}{2|L|}\right)}$$

Finally, let $N = \max\{N_1, N_2\}$

Then for all $n \geq N$,

$|a_n b_n - LM|$

$\leq |b_n||a_n - L| + |L||b_n - M|$

$< |b_n||a_n - L| + \frac{\varepsilon}{2}$

$< \left(|M| + \frac{\varepsilon}{2|L|}\right)|a_n - L| + \frac{\varepsilon}{2}$

$< \cancel{\left(M + \frac{\varepsilon}{2|L|}\right)}\left(\frac{\varepsilon}{2\cancel{\left(M + \frac{\varepsilon}{2|L|}\right)}}\right) + \frac{\varepsilon}{2}$

$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

If $L = 0$ and $M \neq 0$,

interchange the roles of L and M.

$$|a_n b_n - LM|$$

$$= |a_n b_n - a_n M + a_n M - LM|$$

$$< |a_n| |b_n - M| + |M| |a_n - L| .$$

If $L = M = 0$, we want to show

$$|a_n b_n| < \varepsilon$$

Since $a_n \to 0$, $\exists N_1 \in \mathbb{N}$,

$$|a_n| < 1 \quad \forall \ n \geq N_1$$

Since $b_n \to 0$, $\exists N_2 \in \mathbb{N}$,

$$|b_n| < \varepsilon \quad \forall \ n \geq N_2$$

Let $N = \max\{N_1, N_2\}$.

Then $\forall \ n \geq N$,

$$|a_n b_n| = |a_n| \cdot |b_n| < 1 \cdot \varepsilon = \varepsilon$$

4) If $\lim_{n\to\infty} a_n = L$ and

$\lim_{n\to\infty} b_n = M \neq 0$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$$

proof Since $\frac{a_n}{b_n} = a_n \cdot \left(\frac{1}{b_n}\right)$,

the proof is immediate from 3)

provided we know $(b_n \to M)$

$$\Rightarrow \left(\frac{1}{b_n} \to \frac{1}{M}\right)$$

We want

$$\underbrace{\left| \frac{1}{b_n} - \frac{1}{M} \right|}\ < \varepsilon.$$

$$= \left| \frac{M - b_n}{b_n M} \right| < \varepsilon$$

Choose $\gamma > 0$ (to be determined later) and observe that $\exists\ N \in \mathbb{N}$ so that

$$|b_n - M| < \gamma \quad \forall\ n \geq N.$$

If $|b_n - M| < \gamma$, then

$$|M| - |b_n| < \gamma,$$

$$\left(|b_n - M| \geq \big|\, |b_n| - |M| \,\big| \right)$$

we have

$$|M| - \gamma < |b_n|.$$

Hence, $$\frac{1}{|b_n|} < \frac{1}{|M| - \gamma}$$

supposing $|M| - \gamma > 0$.

(if not, switch terms).

Hence,

$$\left|\frac{b_n - M}{b_n M}\right| = |b_n - M| \cdot \frac{1}{|b_n|} \cdot \frac{1}{|M|}$$

$$< \gamma \cdot \frac{1}{|b_n|} \cdot \frac{1}{|M|}$$

$$< \gamma \cdot \frac{1}{|M| - \gamma} \cdot \frac{1}{|M|}$$

$$= \frac{\gamma}{|M|^2 - \gamma |M|} \overset{?}{<} \varepsilon$$

Want $\dfrac{\gamma}{|M|^2 - \gamma|M|} < \varepsilon$, so

$$\gamma < \varepsilon\left(|M|^2 - \gamma|M|\right)$$

$$= \varepsilon|M|^2 - \varepsilon\gamma|M|$$

so

$$\gamma + \varepsilon\gamma|M| < \varepsilon|M|^2$$

$$\gamma(1 + \varepsilon|M|) < \varepsilon|M|^2$$

$$\gamma < \frac{\varepsilon|M|^2}{1 + \varepsilon|M|}$$

Hence, choosing $N \in \mathbb{N}$ so that

$$|b_n - M| < \frac{\varepsilon |M|^2}{1 + \varepsilon |M|},$$

we obtain

$$\left| \frac{1}{b_n} - \frac{1}{M} \right| < \varepsilon \quad \square .$$

Cauchy Sequences

(Section 2.6)

Diverging from book

Recall that a sequence need not be confined to $\mathbb{R}$

If X is any set, a sequence in X is just a function from $\mathbb{N}$ to X.

Definition: (metric space) Let X be a set. A metric on X is a function

$$d : X \times X \to [0, \infty)$$

Satisfying

1) $d(x,x) = 0 \ \forall \ x \in X$ and if $d(x,y)=0$, then $x=y \ \forall \ x,y \in X$

2) $d(x,y) = d(y,x) \ \forall \ x,y \in X$

3) $d(x,z) \le d(x,y) + d(y,z) \ \forall \ x,y,z \in X$ (triangle inequality)

A metric is also called a distance function.

A set X endowed with a metric d is called a metric space.

Examples tomorrow!